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ROBIN FUNCTIONS FOR COMPLEX MANIFOLDS AND APPLICATIONS

Norman Levenberg and Hiroshi Yamaguchi

0. Introduction. In [Y] and later in [LY] the problem of the second variation of the Robin function for a smooth variation of domains in \mathbb{C}^n for $n \geq 2$ was studied. Precisely, let $\mathcal{D} = \cup_{t \in B} (t, D(t)) \subset B \times \mathbb{C}^n$ be a variation of domains $D(t)$ in \mathbb{C}^n each containing a fixed point z_0 and with $\partial D(t)$ of class C^∞ for $t \in B := \{t \in \mathbb{C} : |t| < \rho\}$. We let $g(t, z)$ for $t \in B$ and $z \in \overline{D(t)}$ be the \mathbb{R}^{2n} -Green function for the domain $D(t)$ with pole at z_0 ; i.e., $g(t, z)$ is harmonic in $D(t) \setminus \{z_0\}$, $g(t, z) = 0$ for $z \in \partial D(t)$, and $g(t, z) - \frac{1}{\|z - z_0\|^{2n-2}}$ is harmonic near z_0 . We call

$$\lambda(t) := \lim_{z \rightarrow z_0} [g(t, z) - \frac{1}{\|z - z_0\|^{2n-2}}]$$

the *Robin constant* for $(D(t), z_0)$. Then

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z) \|\nabla_z g\|^2 d\sigma_z - 4c_n \int \int_{D(t)} \sum_{a=1}^n \left| \frac{\partial^2 g}{\partial t \partial \bar{z}_a} \right|^2 dV_z. \quad (1)$$

Here, c_n is a positive dimensional constant and

$$k_2(t, z) := \|\nabla_z \psi\|^{-3} \left[\frac{\partial^2 \psi}{\partial t \partial \bar{t}} \|\nabla_z \psi\|^2 - 2\Re \left\{ \frac{\partial \psi}{\partial t} \sum_{a=1}^n \frac{\partial \psi}{\partial \bar{z}_a} \frac{\partial^2 \psi}{\partial \bar{t} \partial z_a} \right\} + \left| \frac{\partial \psi}{\partial t} \right|^2 \Delta_z \psi \right],$$

where $\psi(t, z)$ is a defining function for \mathcal{D} , is the so-called *Levi-curvature* of $\partial \mathcal{D}$ at (t, z) ; the numerator is the sum of the Levi-form of ψ applied to the n complex tangent vectors $(-\frac{\partial \psi}{\partial z_j}, 0, \dots, \frac{\partial \psi}{\partial \bar{t}}, 0, \dots)$. In particular, if \mathcal{D} is pseudoconvex (strictly pseudoconvex) at a point (t, z) with $z \in \partial D(t)$, it follows that $k_2(t, z) \geq 0$ ($k_2(t, z) > 0$) so that $-\lambda(t)$ is subharmonic in B . Given D a bounded domain in \mathbb{C}^n , we let $\Lambda(z)$ be the Robin constant for (D, z) . If we fix a point $\zeta_0 \in D$, for $\rho > 0$ sufficiently small and $a \in \mathbb{C}^n$, the disk $\{\zeta = \zeta_0 + at, |t| < \rho\} := \zeta_0 + aB$ is contained in D . Under the biholomorphic mapping $T(t, z) = (t, z - at)$ of $B \times D$, we get the variation of domains $\mathcal{D} = T(B \times D)$ where each domain $D(t) := T(t, D) = D - at$ contains ζ_0 . Letting $\lambda(t) = \Lambda(\zeta_0 + at)$ denote the Robin constant for $(D(t), \zeta_0)$ and using (1) yields part of the following result, which was proved in [Y] and [LY].

Theorem. Let D be a bounded pseudoconvex domain in \mathbb{C}^n with C^2 boundary. Then $\log(-\Lambda(z))$ and $-\Lambda(z)$ are real-analytic, strictly plurisubharmonic exhaustion functions for D .

In this note, we study a generalization of the second variation formula (1) to complex manifolds. We use our new formula to develop a “rigidity lemma” which allows us to construct, in certain cases, strictly plurisubharmonic exhaustion functions for Levi-pseudoconvex subdomains D of complex manifolds; i.e., we use the Robin function to verify that D is Stein. We remark that when we use the term *pseudoconvex* in describing certain complex manifolds or domains in complex manifolds, we always mean *Levi-pseudoconvex*.

1. The variation formula. Our general set-up is this: let M be an n -dimensional complex manifold (compact or not) equipped with a Hermitian metric

$$ds^2 = \sum_{a,b=1}^n g_{a\bar{b}} dz_a \otimes d\bar{z}_b$$

and let $\omega := i \sum_{a,b=1}^n g_{a\bar{b}} dz_a \wedge d\bar{z}_b$ be the associated (real) $(1, 1)$ -form. As in the introduction, we take $n \geq 2$. We write $g^{\bar{a}b} := (g_{a\bar{b}})^{-1}$ for the elements of the inverse matrix to $(g_{a\bar{b}})$. Given the standard operators $*, \partial, \bar{\partial}, d = \partial + \bar{\partial}, \delta := -*\partial*$, we get the Laplacian operator

$$\Delta = \delta \bar{\partial} + \bar{\partial} \delta + \bar{\partial} \partial + \partial \bar{\partial}$$

which, in local coordinates acting on functions has the form

$$\Delta u = -2 \left\{ \sum_{a,b=1}^n g^{\bar{b}a} \frac{\partial^2 u}{\partial \bar{z}_b \partial z_a} + \frac{1}{2} \sum_{a,b=1}^n \left(\frac{1}{G} \frac{\partial(Gg^{\bar{b}a})}{\partial z_a} \frac{\partial u}{\partial \bar{z}_b} + \frac{1}{G} \frac{\partial(Gg^{\bar{a}b})}{\partial \bar{z}_a} \frac{\partial u}{\partial z_b} \right) \right\}$$

where $G := \det(g_{a\bar{b}})$. We remark that if ds^2 is Kähler, i.e., if $d\omega = 0$, then $\Delta u = -2 \sum_{a,b=1}^n g^{\bar{b}a} \frac{\partial^2 u}{\partial \bar{z}_b \partial z_a}$.

Given a nonnegative C^∞ function $c = c(z)$ on M , we call a C^∞ function u on an open set $D \subset M$ c -harmonic on D if $\Delta u + cu = 0$ on D . In particular, if we fix a point $p_0 \in M$ and a coordinate neighborhood U of p_0 , we can find a c -harmonic function Q_0 in $U \setminus \{p_0\}$ satisfying

$$\lim_{p \rightarrow p_0} \frac{Q_0(p)}{d(p, p_0)^{2n-2}} = 1$$

where $d(p, p_0)$ is the geodesic distance (with respect to the metric ds^2) between p and p_0 . We call Q_0 a *fundamental solution* for Δ and c at p_0 . Fixing p_0 in a smoothly bounded domain $D \subset M$ and fixing a fundamental solution Q_0 , the c -Green function g for (D, p_0) is the c -harmonic function in $D \setminus \{p_0\}$ satisfying $g = 0$ on ∂D (g is continuous up to ∂D) and $g(p) - Q_0(p)$ is regular at p_0 . We note that, provided $c \not\equiv 0$, the c -Green function *always* exists (cf. [NS]) and is nonnegative on D . Then

$$\lambda := \lim_{p \rightarrow p_0} [g(p) - Q_0(p)]$$

is called the c -Robin constant for (D, p_0) .

Now let $\mathcal{D} = \cup_{t \in B} (t, D(t)) \subset B \times M$ be a variation of domains $D(t)$ in M each containing a fixed point p_0 and with $\partial D(t)$ of class C^∞ for $t \in B$. Let $g(t, z)$ be the c -Green function for $(D(t), p_0)$ and $\lambda(t)$ the corresponding c -Robin constant.

We have

$$\begin{aligned} \frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = & -c_n \int_{\partial D(t)} k_2(t, z) \sum_{a,b=1}^n (g^{\bar{a}b} \frac{\partial g}{\partial \bar{z}_a} \frac{\partial g}{\partial z_b}) d\sigma_z \\ & -4c_n \{ \|\bar{\partial} \frac{\partial g}{\partial t}\|_{D(t)}^2 + \frac{1}{2} \|\sqrt{c} \frac{\partial g}{\partial t}\|_{D(t)}^2 + \int \int_{D(t)} [\Re \{ \frac{1}{i} \frac{\partial g}{\partial t} \bar{\partial} \frac{\partial g}{\partial t} \wedge \partial * \omega \} + \frac{1}{2i} |\frac{\partial g}{\partial t}|^2 \bar{\partial} \partial * \omega] \} \end{aligned}$$

where $\|f\|_{D(t)}^2 = \int_{D(t)} f \wedge * \bar{f} \geq 0$, $d\sigma_z$ is the area element on $\partial D(t)$ with respect to the Hermitian metric, and

$$k_2(t, z) :=$$

$$\left[\sum_{a,b=1}^n g^{\bar{a}b} \frac{\partial \psi}{\partial \bar{z}_a} \frac{\partial \psi}{\partial z_b} \right]^{-3/2} \left[\frac{\partial^2 \psi}{\partial t \partial \bar{t}} \left(\sum_{a,b=1}^n g^{\bar{a}b} \frac{\partial \psi}{\partial \bar{z}_a} \frac{\partial \psi}{\partial z_b} \right) - 2\Re \left\{ \frac{\partial \psi}{\partial t} \left(\sum_{a,b=1}^n g^{\bar{a}b} \frac{\partial \psi}{\partial \bar{z}_a} \frac{\partial^2 \psi}{\partial z_b \partial \bar{t}} \right) \right\} + \left| \frac{\partial \psi}{\partial t} \right|^2 \left(\sum_{a,b=1}^n g^{\bar{a}b} \frac{\partial^2 \psi}{\partial \bar{z}_a \partial z_b} \right) \right],$$

$\psi(t, z)$ being a defining function for \mathcal{D} .

Note that if \mathcal{D} is pseudoconvex at a point $(t, z) \in \partial \mathcal{D}$ with $z \in \partial D(t)$, then $k_2(t, z) \geq 0$. This follows since we can always choose local coordinates near a point $z \in M$ so that $g_{a\bar{b}}(z) = \delta_{ab}$. A simple calculation shows that $\partial * \omega = 0$ if ds^2 is a Kähler metric; hence we have the following result.

Corollary 1.1. Suppose that ds^2 is a Kähler metric on M . Then

$$\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t) = -c_n \int_{\partial D(t)} k_2(t, z) \sum_{a,b=1}^n (g^{\bar{a}b} \frac{\partial g}{\partial \bar{z}_a} \frac{\partial g}{\partial z_b}) d\sigma_z - 4c_n \{ \|\bar{\partial} \frac{\partial g}{\partial t}\|_{D(t)}^2 + \frac{1}{2} \|\sqrt{c} \frac{\partial g}{\partial t}\|_{D(t)}^2 \}. \quad (1')$$

In particular, if \mathcal{D} is pseudoconvex in $B \times M$, then $-\lambda(t)$ is subharmonic on B .

Remark 1. Formula (1') is valid under the weaker assumption that the *complex torsion* of the metric $g_{a\bar{b}}$ vanishes. We do not discuss this notion here. Note that (1') reduces to (1) if $g_{a\bar{b}} = \delta_{ab}$ and $c \equiv 0$.

We consider the same situation as in the corollary. From the variation formula (1') and continuity of $g(t, z)$ up to $\partial D(t)$, we get the following result.

Lemma 1.2 (rigidity). Assume \mathcal{D} is pseudoconvex in $B \times M$, ds^2 is a Kähler metric on M and that there exists $t_0 \in B$ such that $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(t_0) = 0$. If $c(z) \not\equiv 0$ on $D(t_0)$, then

$$\frac{\partial g}{\partial t}(t_0, z) \equiv 0 \text{ on } \overline{D(t_0)}.$$

Remark 2. The same conclusion is valid if we assume that $\partial D(t_0)$ has one strictly pseudoconvex boundary point (instead of (or in addition to) assuming $c(z) \not\equiv 0$ on $D(t_0)$). However, the importance of the above formulation of the rigidity lemma is that, as we will see below, the function c gives us extra flexibility in order to deduce strict pseudonvexity in certain cases.

2. Complex Lie groups. We apply the rigidity lemma to the study of complex Lie groups. Let M be a complex Lie group of complex dimension n with identity e equipped with a Kähler metric ds^2 and let $c = c(z)$ be a nonnegative C^∞ function on M . Let $D \subset M$ be a domain in M with smooth boundary. For $z \in D$, let

$$D(z) := \{wz^{-1} \in D : w \in D\} = D \cdot z^{-1}$$

be right-translation (multiplication) of D by z^{-1} . Note that $D(z)$ is a smoothly bounded domain in M which contains e if $z \in D$; if D and hence $D(z)$ is unbounded, the c -Green function for $(D(z), e)$ can be defined as a limit of c -Green functions for $(D_k(z), e)$ where $\{D_k(z)\}$ are bounded domains with $D_k(z) \subset\subset D_{k+1}(z)$ and $\cup D_k(z) = D(z)$. Let $\Lambda(z)$ denote the c -Robin constant for $(D(z), e)$ (we assume, apriori, that a fundamental solution Q_0 for Δ and c at e is fixed). Our first main result is the following.

Theorem 2.1. Suppose $D \subset\subset M$ is pseudoconvex. Then

1. $-\Lambda(z)$ is a plurisubharmonic exhaustion function for D ;
2. if $c > 0$, then $-\Lambda(z)$ is a strictly plurisubharmonic exhaustion function for D if and only if D is Stein; indeed, if the complex Hessian matrix $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(\zeta)]$ has a zero eigenvalue with (geometric) multiplicity $k \geq 1$ at some point $\zeta \in D$, then the complex Hessian matrix of any plurisubharmonic exhaustion function $s(z)$ for D has a zero eigenvalue with (geometric) multiplicity at least k at each point $z \in D$.

We will sketch the proof of Theorem 2.1. First we remark that there exist n linearly independent left-invariant holomorphic vector fields X_1, \dots, X_n such that $\text{Expt} X_j$, $j = 1, \dots, n$ form local coordinates in a neighborhood V of the identity $e \in M$; then $\zeta \text{Expt} X_j$, $j = 1, \dots, n$ form local coordinates in a neighborhood ζV of $\zeta \in M$. If we fix a direction vector α and consider the complex disk $t \rightarrow \zeta + \alpha t$ for small $|t|$, we can assume that $\zeta \text{Expt} X_1 = \zeta + \alpha t$; for simplicity, we write $X := X_1$. This suggests, as in the variation of domains case described in the introduction, how to set up a variation of domains in the setting of the complex Lie group M . We note, for future use, that $t \rightarrow z \text{Expt} X$ is the unique integral curve to X taking the value $z \in M$ for $t = 0$.

We now let ζ be a fixed point in D and choose $B = \{t \in \mathbb{C} : |t| < \rho\}$ with ρ sufficiently small so that

$$\eta := \zeta \text{Expt} X = \zeta + \alpha t \in D \text{ for all } t \in B. \quad (2)$$

Let $T : B \times M \rightarrow B \times M$ via $T(t, z) = (t, F(t, z)) := (t, w)$ where $w = F(t, z) := z(\zeta \text{Expt} X)^{-1}$. Then $\mathcal{D} := T(B \times D)$ defines a variation of domains $D(t) := F(t, D) = \{z(\zeta \text{Expt} X)^{-1} \in M : z \in D\} = D \cdot (\zeta \text{Expt} X)^{-1}$.

Let $g(t, w)$ be the c -Green function for $(D(t), e)$ and let $\lambda(t) := \Lambda(\zeta \text{Expt} X)$ for $t \in B$; this is the c -Robin constant for $(D(t), e)$ (note $e \in D(t)$ if $t \in B$ by (2)). Then

$$\sum_{j,k=1}^n \frac{\partial^2(-\Lambda)}{\partial \eta_j \partial \bar{\eta}_k}(\zeta) \alpha_j \bar{\alpha}_k = \frac{\partial^2(-\Lambda)}{\partial t \partial \bar{t}}(\zeta \text{Expt} X)|_{t=0} = \frac{\partial^2(-\lambda)}{\partial t \partial \bar{t}}(0). \quad (3)$$

The plurisubharmonicity of $-\Lambda(z)$ now follows from Corollary 1.1 and the fact that $\mathcal{D} := T(B \times D)$ is the biholomorphic image of the pseudoconvex set $B \times D$; indeed, for each $t \in B$, the function $z = \phi(t, w) = (\phi_1(t, w), \dots, \phi_n(t, w)) := w \zeta \text{Expt} X = F^{-1}(t, w)$ is the well-defined holomorphic inverse map of $z \rightarrow w = F(t, z)$ for all $w \in M$. Standard arguments show that $\Lambda(z) \rightarrow -\infty$ as $z \rightarrow z' \in \partial D$ which proves 1. of the theorem.

We will prove 2. in the case where $k = 1$; here, we use the assumption that $c > 0$ and apply the rigidity lemma. The key observation is the following.

Claim: Suppose that $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = 0$.

- a. $z \in D$ (resp. $\partial D, \bar{D}^c$) if and only if $z \text{Expt} X \in D$ (resp. $\partial D, \bar{D}^c$) for all $t \in \mathbb{C}$;
- b. $D \cdot z^{-1} = D \cdot (z \text{Expt} X)^{-1}$ (resp. $\partial D, \bar{D}^c$) for all $t \in \mathbb{C}$ and for each $z \in M$.

To prove the claim, we apply the rigidity lemma to show that the left-invariant holomorphic vector field X is a non-vanishing holomorphic vector field on M satisfying the property that *any integral curve $z(t)$ of X with initial value $X(z_0)$ for $z_0 = z(0) \in \partial D$ remains in ∂D for all $t \in \mathbb{C}$* . This is one implication in part a. of the claim for ∂D .

Recall that $z = \phi(t, w) = (\phi_1(t, w), \dots, \phi_n(t, w)) := w\zeta \text{Expt} X = F^{-1}(t, w)$ for all $w \in M$. Let $t \rightarrow \phi(t, e)$ be the (moving) image under ϕ of the identity element. Note that if $ds_t^2(z)$ denotes the pull-back of the metric $ds^2(w)$ under $F(t, z)$, then the Green function $G(t, z)$ for D with pole at $\phi(t, e)$ (with respect to $ds_t^2(z)$) equals $g(t, w)$. The assumption that $\frac{\partial^2 \lambda}{\partial t \partial \bar{t}}(0) = 0$ yields, by the rigidity lemma, $\frac{\partial g}{\partial t}(0, w) \equiv 0$ for $w \in \bar{D}(0)$; this becomes

$$\frac{\partial G}{\partial t}(0, z) + \sum_{a=1}^n \left[\frac{\partial G}{\partial z_a}(0, z) \frac{\partial \phi_a}{\partial t}(0, F(0, z)) + \frac{\partial G}{\partial \bar{z}_a}(0, z) \frac{\partial \bar{\phi}_a}{\partial t}(0, F(0, z)) \right] = 0$$

for $z \in \bar{D}$. But $\frac{\partial \bar{\phi}_a}{\partial t}(0, F(0, z)) = 0$ since $\phi(t, w) = (\phi_1(t, w), \dots, \phi_n(t, w))$ is holomorphic in t ; and $\frac{\partial G}{\partial t}(0, z) = 0$ for $z \in \partial D$ since $G(t, z) = 0$ for $z \in \partial D$ and $t \in B$. Thus

$$\sum_{a=1}^n \frac{\partial G}{\partial z_a}(0, z) \frac{\partial \phi_a}{\partial t}(0, F(0, z)) = 0 \quad (4)$$

for $z \in \partial D$. Since $\phi(t, w)$ is defined for all $w \in M$, the vector field

$$Y(z) := \sum_{a=1}^n \frac{\partial \phi_a}{\partial t}(0, F(0, z)) \frac{\partial}{\partial z_a}$$

is a globally defined (on M) non-vanishing holomorphic vector field; using the fact that

$$\left(\frac{\partial G}{\partial z_1}(0, z), \dots, \frac{\partial G}{\partial z_n}(0, z) \right)$$

is a (complex) normal vector to ∂D at z , it can be shown that (4) implies that any integral curve $z(t)$ of Y with initial value $Y(z_0)$ for $z_0 = z(0) \in \partial D$ remains in ∂D for all $t \in \mathbb{C}$. Thus, to verify the italicised statement, it suffices to show that $Y = X$.

Since X is left-invariant, if $X(z) = \sum_{a=1}^n \eta_a \frac{\partial}{\partial z_a}$, then $[\frac{\partial}{\partial t}(z \text{Expt} X)_a]|_{t=0} = \eta_a(z)$, $a = 1, \dots, n$. But for $w = z\zeta^{-1}$,

$$\frac{\partial \phi_a}{\partial t}(0, F(0, z)) = \frac{\partial \phi_a}{\partial t}(0, w) = \left[\frac{\partial}{\partial t}(w\zeta \text{Expt} X)_a \right]|_{t=0} = \eta_a(w\zeta) = \eta_a(z),$$

which gives the result.

The proof of the claim is now immediate. For example, to establish a. for ∂D ; i.e., to show $z \in \partial D$ if and only if $z \text{Expt} X \in \partial D$ for all $t \in \mathbb{C}$, the “only if” direction has already been proved. Suppose now that $z \text{Expt} X \in \partial D$ for all $t \in \mathbb{C}$. Since

$$z = z(\text{Expt} X)(\text{Exp}(-tX)) := z' \text{Exp}(-tX)$$

where $z' = z \text{Expt} X \in \partial D$, the previous argument shows that $z \in \partial D$. Since ∂D is a smooth, closed $(2n-1)$ -dimensional real hypersurface in M , the analogous results for D and \bar{D}^c follow from uniqueness of the integral curve $t \rightarrow z \text{Expt} X$. Similarly we prove b. only for ∂D . Let $z_1 \in \partial D$ and $z \in M$. Since $z_1 \text{Expt} X \in \partial D$ for all $t \in \mathbb{C}$ from a. of the claim, the equation

$$z_1 z^{-1} = z_1(\text{Expt} X)(\text{Exp}(-tX))z^{-1} = z_1(\text{Expt} X)(z \text{Expt} X)^{-1}$$

yields b. of the claim.

We can now finish the proof of 2. of the theorem. For a point $\zeta \in D$, let $a_i(\zeta)$, $i = 1, \dots, n$ denote the eigenvalues of $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(\zeta)]$ at ζ . To prove 2. in the case $k = 1$, we suppose there exists a point $\zeta \in D$ with $a_1(\zeta) = 0$; without loss of generality, we can assume that $\zeta \text{Expt} X_1 = \zeta + \alpha t$ gives the direction of the corresponding eigenvector; i.e.,

$$\frac{\partial^2(-\Lambda)}{\partial t \partial \bar{t}}(\zeta + \alpha t)|_{t=0} = 0. \quad (5)$$

Taking $X = X_1$ in the previous claim, $D_1(t) := D \cdot (\zeta \text{Expt} X_1)^{-1}$ and $\lambda_1(t) := \Lambda(\zeta \text{Expt} X_1)$, (5) becomes $\frac{\partial^2(-\lambda_1)}{\partial t \partial \bar{t}}(0) = 0$. Then the integral curve $t \rightarrow z \text{Expt} X_1$, $t \in \mathbb{C}$, satisfies the conditions of the claim. In particular, if $z \in D$, then $D \cdot z^{-1} = D \cdot (z \text{Expt} X_1)^{-1}$ for all $t \in \mathbb{C}$ which implies that

$$\Lambda(z \text{Expt} X_1) = \Lambda(z) \text{ for all } t \in \mathbb{C}.$$

But $-\Lambda$ is an exhaustion function for D ; hence the image C_z of the integral curve $t \rightarrow z \text{Expt} X_1$, $t \in \mathbb{C}$ is compactly contained in D and $-\Lambda$ is constant on C_z . In particular, $-\Lambda$ is *harmonic* on C_z for each $z \in D$; i.e., $[\frac{\partial^2(-\Lambda)}{\partial z_j \partial \bar{z}_k}(z)]$ has a zero eigenvalue $a_1(z)$ for each $z \in D$. But then if s is *any* plurisubharmonic exhaustion function for D , s is also subharmonic and entire on each complex curve C_z and hence constant (and harmonic) on this curve, which implies that $[\frac{\partial^2 s}{\partial z_j \partial \bar{z}_k}(z)]$ has a zero eigenvalue for each $z \in D$. In particular, D is not Stein.

Remark 3. Note that if M is a Stein manifold, then each pseudoconvex $D \subset\subset M$ is Stein; this occurs, for example, if M is a simply connected solvable Lie group or if M is connected and semi-simple (cf. [GR]).

3. Complex homogeneous spaces. In this section, we let M be a complex space with the property that there exists a complex Lie group $G \subset \text{Aut} M$ of complex dimension n which acts transitively on M . As prototypical examples, we can take $M = \mathbb{P}^N$ = complex projective space, or, more generally, we can take $M = G(k, N)$ = complex Grassmann manifold (and $G = \text{Aut} M$). Let $D \subset\subset M$ be a domain with smooth boundary. For $z \in M$, we let

$$D(z) := \{g \in G : g(z) \in D\}$$

be a (possibly unbounded) domain in G . Note that if $z \in D$, then the identity element e of G lies in $D(z)$. Thus if we let ds^2 be a Kähler metric on G and let c be a nonnegative smooth function on G , we can form the c -Robin constant $\lambda(z)$ for $(D(z), e)$ (recall that the c -Green function is defined by the usual exhaustion method for unbounded domains). Using the ideas and techniques from the previous section, we can prove the following result.

Theorem 3.1. Suppose D is pseudoconvex in M . Then for $z \in D$, $D(z)$ is pseudoconvex in G and $-\lambda(z)$ is a plurisubharmonic exhaustion function for D . Furthermore, if $c > 0$ in G and G is doubly transitive on M , then $-\lambda(z)$ is strictly plurisubharmonic; i.e., D is Stein.

Recall that G is *doubly transitive* on M if for pairs of points $(a, b), (c, d) \in M$, there exists $g \in G$ with $g(a) = c$ and $g(b) = d$. This is equivalent to the *three point property* of (M, G) : for each triple of points $a, b, c \in M$, there exists $g \in G$ with $g(a) = a$ and $g(b) = c$. Details of the proof of Theorem 3.1 will be given elsewhere.

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